Unsteady MHD flow of a Couple Stress Fluid Through a Porous Medium Between Parallel Plates Under the Influence of Pulsation of Pressure Gradient

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Abstract— In this chapter, we discuss an analytical study of unsteady magneto hydro dynamic flow of an incompressible electrically conducting couple stress fluid through a porous medium between parallel plates, taking into account pulsation of the pressure gradient effect and under the influence of a uniform inclined magnetic field of strength H_o inclined at an angle of inclination α with the normal to the boundaries. The solution of the problem is obtained with the help of perturbation technique. Analytical expression is given for the velocity field and the effects of the various governing parameters entering into the problem are discussed with the help of graphs. The shear stresses on the boundaries and the discharge between the plates are also obtained analytically and their behaviour computationally discussed with different variations in the governing parameters in detail.

Index Terms— couple stress fluids, inclined magnetic field, magneto hydro dynamic flows, porous medium, pulsation of the pressure gradient, parallel plate channels and unsteady flows

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1 INTRODUCTION

A fluid flow driven by a pulsatile pressure gradient through porous media is of great interest in physiology and Biomedical Engineering. Such a study has application in the dialysis of blood through artificial kidneys or blood flow in the lung alveolar sheet. Ahmadi and Manvi [2] derived a general equation of motion for flow through porous medium and applied it to some fundamental flow problems. Rapits [8] has studied the flow of a polar fluid through a porous medium, taking angular velocity into account. The problem of peristaltic transport in a cylindrical tube through a porous medium has been investigated by El-Shehawey and El-Sebaei [7], their results show that the fluid phase means axial velocity increases with increasing the permeability parameter k . Afifi and Gad [1] have studied the flow of a Newtonian, incompressible fluid under the effect of transverse magnetic field through a porous medium between infinite parallel walls on which a sinusoidal traveling wave is imposed. The flow characteristics of a Casson fluid in a tube filled with a homogenous porous medium was investigated by Dash et al [6]. Bhuyan Hazarika [4] has studied the pulsatile flow of blood in a porous channel in the presence of transverse magnetic field. The flows in bends and branches are of interest in a physiological context for several reasons. The additional energy losses due to the local disturbances of the flow are of interest in calculating the air flow in the lungs and in wave-propagation models of the arterial system.

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The details of the pressure and shear stress distribution on the walls of a bend or bifurcation are of interest in the study of parthenogenesis because it appears that the localization of plaques is related to the local flow patterns. In vascular surgery questions arise, such as what is the best angle for vascular graft to enter an existing artery in a coronary bypass (Skalak, R. and Nihat Ozkaya, [12]). The theory of laminar, steady one-dimensional gravity flow of a non-Newtonian fluid along a solid plane surface for a fluid exhibiting slope at the wall has been studied by Astarita et al [3]. Suzuki and Tanaka [13] have carried out some experiments on non-Newtonian fluid along an inclined plane, the flow of Rivlin-Ericksen incompressible fluid through an inclined channel with two parallel flat walls under the influence of magnetic field has been studied by Rathod and Shrikanth [11]. Rathod and Shrikanth [9] have studied the MHD flow of Rivlin-Ericksen fluid between two infinite parallel inclined plates. The gravity flow of a fluid with couple stress along an inclined plane at an angle with horizontal has been studied by Chaturani and Upadhya [5]. Rathod and Thippeswamy [10] have studied the pulsatile flow of blood through a closed rectangular channel in the presence of microorganisms for gravity flow along an inclined channel. Hence, it appears that inclined plane is a useful device to study some properties of non-Newtonian fluids.

In this paper, we discuss an analytical study of unsteady magneto hydro dynamic flow of an incompressible electrically conducting couple stress fluid through a porous medium between parallel plates, taking into account pulsation of the pressure gradient effect and under the influence of a uniform inclined magnetic field of strength H_o inclined at an angle of inclination α with the normal to the boundaries.

2 FORMULATION AND SOLUTION OF THE PROBLEM

We consider the unsteady hydro magnetic flow of a couple stress fluid through a porous medium induced by the pulsation of the pressure gradient. The plates are assumed to be electrically insulated. The fluid is driven by a uniform pressure gradient parallel to the channel plates and the entire flow field is subjected to a uniform inclined magnetic field of strength H_o inclined at an angle of inclination α with the normal to the boundaries in the transverse xy-plane.

We choose a Cartesian system O(x, y, z) such that the boundary walls are at $z=0$ and $z=l$ and are assumed to be parallel to xy-plane. The equations for steady flow through porous medium are governed by Brinkman's model. At the interface the fluid satisfies the continuity condition of velocity and stress. The boundary plates are assumed to be parallel to xy-plane and the magnetic field of strength H_o inclined at an angle of inclination α to the z-axis in the transverse xz-plane. This inclined magnetic field on the axial flow along the x-direction gives rise to the current density along y-direction in view of Ohm's law. Also the inclined magnetic field in the presence of current density exerts a Lorentz force with components along O(x, z) direction, The component along z-direction induces a secondary flow in that direction while its x-components changes perturbation to the axial flow.

The steady hydro magnetic equations governing the couple stress fluid under the influence of a uniform inclined magnetic field of strength H_o inclined at an angle of inclination α with reference to a frame are,

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial z^2} - \frac{\eta}{\rho}\frac{\partial^4 u}{\partial z^4} - \frac{\sigma \mu_e^2 H_0^2 Sin^2\alpha}{\rho}u - \frac{\mu}{k\rho}u \quad (1)$$

$$\frac{\partial w}{\partial t} = \frac{\mu}{\rho}\frac{\partial^2 u}{\partial z^2} - \frac{\eta}{\rho}\frac{\partial^4 u}{\partial z^4} - \frac{\sigma \mu_e^2 H_0^2 Sin^2\alpha}{\rho}u - \frac{\mu}{k\rho}u \quad (2)$$

Where, the term $-\frac{\eta}{\rho}\frac{\partial^4 u}{\partial z^4}$ in the above equation gives the effect of couple stresses. All the physical quantities in the above equation have their usual meaning. (u, w) are the velocity components along O(x, z) directions respectively. ρ is the density of the fluid, μ_e is the magnetic permeability, ν is the coefficient of kinematic viscosity, k is the permeability of the medium, H_o is the applied magnetic field.

Let $q = u + iw$

Now combining the equations (1) and (2), we obtain

$$\frac{\partial q}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 q}{\partial z^2} - \frac{\eta}{\rho}\frac{\partial^4 q}{\partial z^4} - \frac{\sigma \mu_e^2 H_0^2 Sin^2\alpha}{\rho}q - \frac{\mu}{k\rho}q \quad (3)$$

The boundary conditions are, (Since the couple stresses vanish at both the plates which in turn) implies that

$$q = 0 \quad , \quad at \quad z = 0 \quad (4)$$
$$q = 0 \quad , \quad at \quad z = l \quad (5)$$

$$\frac{d^2 q}{dz^2} = 0 , \quad at \quad z = 0 \quad (6)$$

$$\frac{d^2 q}{dz^2} = 0 , \quad at \quad z = l \quad (7)$$

We introduce the non-dimensional variables

$$z^* = \frac{z}{l}, \ q^* = \frac{ql}{\nu}, P^* = \frac{Pl^2}{\rho\nu^2}, t^* = \frac{t\nu}{l^2}, \omega^* = \frac{\omega l^2}{\nu}, \ x^* = \frac{x}{l}.$$

Using the non-dimensional variables (dropping asterisks), we obtain

$$a^2\frac{\partial q}{\partial t} + \frac{\partial^4 q}{\partial z^4} - a^2\frac{\partial^2 q}{\partial z^2} + (M^2 Sin^2\alpha + D^{-1})a^2 q = -a^2\frac{\partial p}{\partial x} \quad (8)$$

Where $\quad a^2 = \dfrac{l^2\mu}{\eta}\quad$ is the couple stress parameter

$\quad M^2 = \dfrac{\sigma \mu_e^2 H_0^2 l^2}{\mu}\quad$ is the Hartman number

$\quad D^{-1} = \dfrac{l^2}{k}\quad$ is the inverse Darcy parameter

Corresponding the non-dimensional boundary conditions are given by

$$q = 0 \quad , \quad at \quad z = 0 \quad (9)$$
$$q = 0 \quad , \quad at \quad z = 1 \quad (10)$$

$$\frac{d^2 q}{dz^2} = 0 , \quad at \quad z = 0 \quad (11)$$

$$\frac{d^2 q}{dz^2} = 0 , \quad at \quad z = 1 \quad (12)$$

For the pulsation pressure gradient

$$-\frac{\partial p}{\partial x} = \left(\frac{\partial p}{\partial x}\right)_s + \left(\frac{\partial p}{\partial x}\right)_o exp(i\omega t) \quad (13)$$

Equation (6.2.8) reduces to the form

$$a^2\frac{\partial q}{\partial t} + \frac{\partial^4 q}{\partial z^4} - a^2\frac{\partial^2 q}{\partial z^2} + (M^2 Sin^2\alpha + D^{-1})a^2 q =$$

$$-a^2\left\{\left(\frac{\partial p}{\partial x}\right)_s + \left(\frac{\partial p}{\partial x}\right)_o e(i\omega t)\right\} \quad (14)$$

The equation (14) can be solved by using the following perturbation technique

$$u = u_s + u_o e(i\omega t) \quad (15)$$

Substituting the equation (15) in (14) and equating like terms on both sides

$$\frac{d^4 q_s}{dz^4} - a^2\frac{d^2 q_s}{dz^2} + (M^2 Sin^2\alpha + D^{-1})a^2 q_s = -a^2\left(\frac{\partial p}{\partial x}\right)_s \quad (16)$$

and

$$\frac{d^4 q_o}{dz^4} - a^2\frac{d^2 q_o}{dz^2} + (M^2 Sin^2\alpha + D^{-1} + i\omega)a^2 q_o = -a^2\left(\frac{\partial p}{\partial x}\right)_o \quad (17)$$

Subjected to the boundary conditions

$$q_s = 0 \quad , \qquad at \qquad z = 0 \qquad\qquad (18)$$

$$q_s = 0 \quad , \qquad at \qquad z = 1 \qquad\qquad (19)$$

$$\frac{d^2 q_s}{dz^2} = 0 , \qquad at \qquad z = 0 \qquad\qquad (20)$$

$$\frac{d^2 q_s}{dz^2} = 0 , \qquad at \qquad z = 1 \qquad\qquad (21)$$

and

$$q_o = 0 \quad , \qquad at \qquad z = 0 \qquad\qquad (22)$$

$$q_o = 0 \quad , \qquad at \qquad z = 1 \qquad\qquad (23)$$

$$\frac{d^2 q_o}{dz^2} = 0 , \qquad at \qquad z = 0 \qquad\qquad (24)$$

$$\frac{d^2 q_o}{dz^2} = 0 , \qquad at \qquad z = 1 \qquad\qquad (25)$$

$$\left(\frac{\partial p}{\partial x}\right)_s = p_s \quad \text{and} \quad \left(\frac{\partial p}{\partial x}\right)_o = p_o$$

The solutions of the equations (16) and (17) subjected to the boundary conditions (18) to (25) give the velocity distribution of the fluid under consideration.

$$q = C_1 e^{m_1 z} + C_2 e^{m_2 z} + C_3 e^{-m_1 z} + C_4 e^{-m_2 z} + \frac{p_s + G\cos\phi}{M^2 Sin^2\alpha + D^{-1}} +$$

$$+ \left(C_5 e^{m_5 y} + C_6 e^{m_6 y} + C_7 e^{-m_5 y} + C_8 e^{-m_6 y} + \right.$$

$$\left. + \frac{p_o}{M^2 Sin^2\alpha + D^{-1} + i\omega} \right) exp(i\omega t) \qquad (26)$$

Where, the constants $C_1, C_2, \ldots\ldots C_8$ are given in appendix. The shear stresses on the lower and upper plates are given in dimension less form as

$$\tau_L = \left(\frac{dq}{dz}\right)_{z=0}$$

$$= m_1(C_1 - C_3) + m_2(C_2 - C_4)$$

$$+ \left(m_5(C_5 - C_7) + m_6(C_6 - C_8)\right) exp(i\omega t) \qquad (27)$$

And

$$\tau_U = \left(\frac{dq}{dz}\right)_{z=1}$$

$$= m_1(C_1 e^{m_1} - C_3 e^{-m_1}) + m_2(C_2 e^{m_2} - C_4 e^{-m_2}) +$$

$$+ \left(m_5(C_5 e^{m_5} - C_7 e^{-m_5}) + m_6(C_6 e^{m_6} - C_8 e^{-m_6})\right) exp(i\omega t) \qquad (28)$$

The non-dimensional discharge between the plates per unit depth is given by Q

$$Q = \int_0^1 q(z,t) dz$$

$$= \frac{C_1}{m_1}(e^{m_1} - 1) + \frac{C_2}{m_2}(e^{m_2} - 1) - \frac{C_3}{m_1}(e^{-m_1} - 1) - \frac{C_4}{m_2}(e^{-m_1} - 1)$$

$$+ \frac{p_s}{M^2 Sin^2\alpha + D^{-1}} +$$

$$\left(\frac{C_5}{m_5}(e^{m_5} - 1) + \frac{C_6}{m_6}(e^{m_6} - 1) - \frac{C_7}{m_5}(e^{-m_5} - 1) - \frac{C_8}{m_6}(e^{-m_6} - 1) \right.$$

$$\left. + \frac{p_o}{M^2 Sin^2\alpha + D^{-1} + i\omega} \right) exp(i\omega t) \qquad (29)$$

3 RESULTS AND DISCUSSIONS

The unsteady state velocities representing the ultimate flow have been computed numerically for different sets of governing parameters namely viz. The Hartmann parameter M, the inverse Darcy parameter D^{-1} and couple stress parameter a and their profiles are plotted in figures (1-3) and (4-6) for the velocity components u and v respectively. For computational purpose we have assumed an angle of inclination α and the pulsation of pressure gradient in the x-direction and are fixed. Since the thermal buoyancy balances the pressure gradient in the absence of any other applied force in the direction, the flow takes place in planes parallel to the boundary plates. However the flow is three dimensional and all the perturbed variables have been obtained using boundary layer type equations, which reduce to two coupled differential equations for a complex velocity.

We notice that the magnitude of the velocity component u reduces and v increases with increasing the intensity of the magnetic field M the other parameters being fixed, it is interesting to note that the resultant velocity experiences retardation with increasing M (Fig. 1 & 4). (Fig. 2 & 5) exhibit both the velocity components u and v reduces with increasing the inverse Darcy parameter D^{-1}. Lower the permeability of the porous medium lesser the fluid speed in the entire fluid region. The resultant velocity experiences retardation with increasing the inverse Darcy parameter D^{-1}. Here we observe that the retardation due to an increase in the porous parameter is more rapid than that due to increase in the Hartmann number M. In other words, the resistance offered by the porosity of the medium is much more than the resistance due to the magnetic lines of force. We notice that u exhibits a great enhancement in contrast to v which retards appreciably with increase in the couple stress parameter S, but the resultant velocity shows and appreciable enhancement with in a (Fig. 3 & 6).

The shear stresses on the upper and lower plates and the discharge between the plates are calculated computationally and tabulated in the tables (1-5). The magnitude of these stresses at the upper plate is very high compared to the respective magnitudes at the lower plate. We notice that the magnitude of the both stresses τ_x and τ_y increase with increasing the couple stress parameter a on the upper plate and lower plates. On the upper plate, the magnitudes of τ_x and

τ_y increase with increasing M, but τ_x reduces and τ_y enhances with increase in D^{-1}, while on the lower plate τ_y rapidly enhances and τ_x reduces with increase in M. The reversal behavior shows that τ_x and τ_y with increase in D^{-1} (Tables. 1-4). The discharge Q reduces in general with increase in the intensity of the magnetic field M and lower permeability of the porous medium (corresponding to an increase in D^{-1}) and enhances the couple stress parameter a (Table. 5).

4 CONCLUSIONS

Under the effect of pulsation of pressure gradient, the resultant velocity experiences retardation with increasing M, where as the resultant velocity experiences retardation with increasing the inverse Darcy parameter D^{-1}. When we increase the couple stress fluid parameter, the resultant velocity shown and appreciable enhancement in the entire flow region. The magnitude of these stresses at the upper plate is very high compared to the respective magnitudes at the lower plate. The discharge Q reduces in general with increase in the intensity of the magnetic field M and lower permeability of the porous medium and enhances the couple stress parameter a

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TABLE 1
The shear stresses (τ_x) on the upper plate.

a²	I	II	III	IV	V
0.25	1.085652	1.226565	1.566532	0.995682	0.457562
0.5	1.468656	1.566825	1.856825	1.246824	0.675645
0.75	1.533685	1.994526	2.085662	1.327564	0.766528
1	1.636899	2.246535	2.856824	1.457562	0.899465

	I	II	III	IV	V
M	2	5	8	2	2
D^{-1}	1000	1000	1000	2000	3000

TABLE 2
The shear stresses (τ_y) on the upper plate.

a²	I	II	III	IV	V
0.25	-0.45362	-0.63462	-0.85666	-0.66524	-1.56652
0.5	-0.76832	-0.79965	-0.99453	-0.94535	-2.00856
0.75	-0.99526	-1.00855	-1.45665	-1.47565	-2.24656
1	-1.28656	-0.38408	-1.88457	-1.83465	-2.56658

	I	II	III	IV	V
M	2	5	8	2	2
D^{-1}	1000	1000	1000	2000	3000

TABLE 3
The shear stresses (τ_x) on the lower plate.

a²	I	II	III	IV	V
0.25	0.028465	0.023265	0.018345	0.045652	0.065652
0.5	0.036652	0.033457	0.030085	0.095658	0.256656
0.75	0.042754	0.039942	0.032683	0.146524	0.846834
1	0.051156	0.045768	0.035683	0.284682	0.984653

	I	II	III	IV	V
M	2	5	8	2	2
D^{-1}	1000	1000	1000	2000	3000

TABLE 4
The shear stresses (τ_y) on the lower plate.

a²	I	II	III	IV	V
0.25	-0.03265	-0.03846	-0.04256	-0.01561	-0.00566
0.5	-0.04211	-0.04808	-0.05245	-0.02564	-0.01579
0.75	-0.05754	-0.06334	-0.06856	-0.04751	-0.02668
1	-0.06224	-0.06568	-0.07566	-0.05996	-0.03468

	I	II	III	IV	V
M	2	5	8	2	2
D^{-1}	1000	1000	1000	2000	3000

Table 5
Discharge

a²	I	II	III	IV	V
0.25	0.568898	0.502142	0.455789	0.487988	0.411254
0.5	0.685998	0.488755	0.388547	0.558264	0.501245
0.75	0.478871	0.322565	0.244587	0.445879	0.410025
1	0.356658	0.255478	0.189965	0.312455	0.322546

	I	II	III	IV	V
M	2	5	8	2	2
D^{-1}	1000	1000	1000	2000	3000

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Appendices

$$m_1 = \sqrt{\frac{a^2 + \sqrt{a^4 - 4a^2(M^2 Sin^2\alpha + D^{-1})}}{2}}$$

$$m_2 = \sqrt{\frac{a^2 - \sqrt{a^4 - 4a^2(M^2 Sin^2\alpha + D^{-1})}}{2}}$$

$$m_5 = \sqrt{\frac{a^2 + \sqrt{a^4 - 4a^2(M^2 Sin^2\alpha + D^{-1} + i\omega)}}{2}}$$

$$m_6 = \sqrt{\frac{a^2 - \sqrt{a^4 - 4a^2(M^2 Sin^2\alpha + D^{-1} + i\omega)}}{2}}$$

$$C_1 = -\left\{ C_2 + C_3 + C_4 + \frac{p_s}{M^2 Sin^2\alpha + D^{-1}} \right\}$$

$$C_2 = \frac{-1}{e^{m_2} - e^{m_1}} \left\{ \begin{array}{l} \left(e^{-m_1} - e^{m_1}\right)C_3 + \left(e^{-m_2} - e^{m_1}\right)C_4 + \\ + \left(\frac{p_s}{M^2 Sin^2\alpha + D^{-1}}\right)\left(1 - e^{m_1}\right) \end{array} \right\}$$

$$C_3 = -\frac{d_2}{d_1}C_4 - \frac{d_3}{d_1}, \quad C_4 = \frac{d_3 d_4 - d_1 d_6}{d_1 d_5 - d_2 d_4}$$

$$C_5 = -\left\{ C_6 + C_7 + C_8 + \frac{p_o}{M^2 Sin^2\alpha + D^{-1} + i\omega} \right\}$$

$$C_6 = \frac{-1}{e^{m_6} - e^{m_5}} \left\{ \begin{array}{l} \left(e^{-m_5} - e^{m_5}\right)C_7 + \left(e^{-m_6} - e^{m_5}\right)C_8 + \\ \left(\frac{p_o}{M^2 Sin^2\alpha + D^{-1} + i\omega}\right)\left(1 - e^{m_5}\right) \end{array} \right\}$$

$$C_7 = -\frac{d_2'}{d_1'}C_8 - \frac{d_3'}{d_1'}, \quad C_8 = \frac{d_3' d_4' - d_1' d_6'}{d_1' d_5' - d_2' d_4'}$$

$$d_1 = \frac{(m_2^2 - m_1^2)(e^{m_1} - e^{-m_1})}{(e^{m_2} - e^{m_1})}$$

$$d_2 = (m_2^2 - m_1^2)\left\{ \frac{(e^{m_2} - e^{-m_2})}{(e^{m_2} - e^{m_1})} \right\}$$

$$d_3 = \frac{p_s}{M^2 Sin^2\alpha + D^{-1}} \left\{ \frac{(m_1^2 - m_2^2)(1 - e^{m_1})}{(e^{m_2} - e^{m_1})} - m_1^2 \right\}$$

$$d_4 = \frac{e^{m_2}(m_1^2 - m_2^2)(e^{-m_1} - e^{m_1})}{(e^{m_2} - e^{m_1})}$$

$$d_5 = \frac{(m_2^2 e^{-m_2} - m_1^2 e^{m_1})(e^{m_2} - e^{m_1}) - (e^{-m_2} - e^{m_1})(m_2^2 e^{m_2} - m_1^2 e^{m_1})}{(e^{m_2} - e^{m_1})}$$

$$d_6 = \frac{p_s}{M^2 Sin^2\alpha + D^{-1}} \left\{ \frac{(e^{m_2} - e^{m_1})(-m_1^2 e^{m_1}) - (1 - e^{m_1})(m_2^2 e^{m_2} - m_1^2 e^{m_1})}{(e^{m_2} - e^{m_1})} \right\}$$

$$d_1' = \frac{(m_6^2 - m_5^2)(e^{m_5} - e^{-m_5})}{(e^{m_6} - e^{m_5})}, \quad d_2' = (m_6^2 - m_5^2)\left\{ \frac{(e^{m_6} - e^{-m_6})}{(e^{m_6} - e^{m_5})} \right\}$$

$$d_3' = \frac{p_o}{M^2 Sin^2\alpha + D^{-1} + i\omega} \left\{ \frac{(m_5^2 - m_6^2)(1 - e^{m_5})}{(e^{m_6} - e^{m_5})} - m_5^2 \right\}$$

$$d_4' = \frac{e^{m_6}(m_5^2 - m_6^2)(e^{-m_5} - e^{m_5})}{(e^{m_6} - e^{m_5})}$$
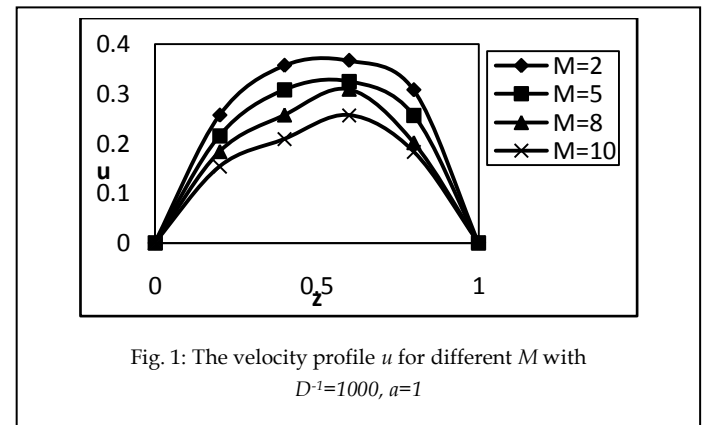
GRAPHS

Fig. 1: The velocity profile u for different M with
$D^{-1}=1000$, a=1

Fig. 2: The velocity profile *u* for different D^{-1} with
M=2, a=1

Fig. 5: The velocity profile *v* for different D^{-1} with
M=2, a=1

Fig. 3: The velocity profile *u* for different *a* with
$D^{-1}=1000, M=2$

Fig. 6: The velocity profile *v* for different *a* with
$D^{-1}=1000, M=2$

Fig. 4: The velocity profile *v* for different *M* with
$D^{-1}=1000, a=1$